Photon–graviton mixing in an electromagnetic field

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Abstract
Einstein–Maxwell theory implies the mixing of photons with gravitons in an external electromagnetic field. This process and its possible observable consequences have been studied at tree level for many years. We use the worldline formalism for obtaining an exact integral representation for the one-loop corrections to this amplitude due to scalars and fermions. We study the structure of this amplitude, and obtain exact expressions for various limiting cases.

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1. Photon–graviton mixing at tree level and one loop

As has been realized already in the 1960s [1] Einstein–Maxwell theory in a constant electromagnetic field contains a tree level vertex for photon–graviton conversion,

$$\kappa h_{\mu\nu} F^{\mu\alpha} f^{\nu}{}_{\alpha} - \tfrac{1}{4}\kappa h^{\mu}_{\mu} F^{\alpha\beta} f_{\alpha\beta}. \tag{1}$$

Here $h_{\mu\nu}$ denotes the graviton, $f_{\mu\nu}$ is the photon and $F^{\mu\nu}$ is the external field. κ is the gravitational coupling constant. This vertex implies the possibility of photon–graviton oscillations [1–7] which are analogous to the better-known photon–axion oscillations in a field [8]. In the presence of an external field, the true eigenstates of propagation can be obtained by solving the following system of dispersion relations [3]:

$$\begin{pmatrix} \eta^{\alpha\beta}k^2 - k^\alpha k^\beta & \tfrac{i}{2}\kappa C^{\kappa\lambda,\alpha} \\ \tfrac{i}{2}\kappa C^{\mu\nu,\beta} & \tfrac{k^2}{4}(\eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa} - 2\eta^{\mu\nu}\eta^{\kappa\lambda} + ...) \end{pmatrix} \begin{pmatrix} a_\beta(k) \\ h_{\kappa\lambda}(k) \end{pmatrix} = 0. \tag{2}$$

Here $C^{\mu\nu,\alpha}$ denotes the Fourier transform of the vertex (1),

$$C^{\mu\nu,\alpha} = (F \cdot k)^\alpha \eta^{\mu\nu} + F^{\mu\alpha}k^\nu + F^{\nu\alpha}k^\mu - (F \cdot k)^\mu \eta^{\nu\alpha} - (F \cdot k)^\nu \eta^{\mu\alpha}. \tag{3}$$

For small deviations from the vacuum dispersion relations $k^2 = 0$, this second order equation can be linearized. An efficient formalism for solving the system (3) under this condition was developed in [3].

Figure 1. One-loop photon–graviton amplitude in a constant field. The double line represents the propagator of a charged scalar or spin $\frac{1}{2}$ particle in a constant field.

Taking one-loop corrections into account, the dispersion relation matrix gets modified in the following way [9, 10],

$$\begin{pmatrix} \eta^{\alpha\beta}k^2 - k^\alpha k^\beta - \bar{\Pi}^{\alpha,\beta} & \frac{\mathrm{i}}{2}\kappa C^{\kappa\lambda,\alpha} - \bar{\Pi}^{\kappa\lambda,\alpha} \\ \frac{\mathrm{i}}{2}\kappa C^{\mu\nu,\beta} - \bar{\Pi}^{\mu\nu,\beta} & \frac{k^2}{4}(\eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa} - 2\eta^{\mu\nu}\eta^{\kappa\lambda} + ...) - \bar{\Pi}^{\mu\nu,\kappa\lambda} \end{pmatrix} \begin{pmatrix} a_\beta(k) \\ h_{\kappa\lambda}(k) \end{pmatrix} = 0.$$

(4)

Here $\bar{\Pi}^{\alpha,\beta}$, $\bar{\Pi}^{\mu\nu,\beta}$ and $\bar{\Pi}^{\mu\nu,\kappa\lambda}$ denote the one-loop photon–photon, graviton–photon and graviton–graviton vacuum polarization tensors in a constant field. These quantities in principle have to be calculated with all possible loop particles. Equations (4) generalize the QED dispersion relation

$$(\eta^{\alpha\beta}k^2 - k^\alpha k^\beta - \bar{\Pi}^{\alpha,\beta})a_\beta(k) = 0.$$

(5)

This case is well known and has been studied by many authors (see, e.g., [11–13]). It leads to a complicated dependence of the phase velocity on polarization, field strength and frequency (see [14] for a detailed discussion).

In this paper, we report on our recently concluded calculation [9, 10] of the photon–graviton polarization tensor in a constant electromagnetic field $\bar{\Pi}^{\mu\nu,\alpha}$, with a charged scalar or spin $\frac{1}{2}$ particle in the loop. As a Feynman diagram, this amplitude is represented by figure 1.

In [9] the worldline formalism was used to obtain compact parameter integral representations for this amplitude. The numerical and structural analysis has been concluded only recently [10]. Since this formalism is presently still somewhat novel, particularly in applications to gravity, we will start with briefly reviewing its basics from a user's point of view.

2. Worldline formalism in a constant electromagnetic field

The worldline formalism goes back to Feynman's representation of scalar [15] and spinor [16] QED in terms of relativistic particle path integrals. Let us write down Feynman's integral for the simplest possible case, the one-loop effective action in scalar QED:

$$\Gamma(A) = \int \mathrm{d}^4 x \mathcal{L}(A) = \int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, \mathrm{e}^{-S[x(\tau)]}.$$

(6)

Here m and T denote the mass and proper time of the loop scalar, respectively. The worldline path integral is to be calculated over loops in spacetime with a fixed periodicity T and a worldline action given by

$$S[x(\tau)] = \int_0^T \mathrm{d}\tau \left[\frac{1}{4}\dot{x}^2 + \mathrm{i}e\dot{x}^\mu A_\mu(x(\tau)) \right].$$

(7)

In the fermion QED case, a number of different ways have been found to implement the spin in the worldline path integral. Feynman's original formulae [16] are based on a spin factor

involving Dirac matrices, but for the purpose of analytic calculation it is usually preferable to implement spin by the following additional Grassmann path integral [17],

$$\int \mathcal{D}\psi(\tau) \exp\left[-\int_0^T d\tau \left(\frac{1}{2}\psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right]. \tag{8}$$

Here the functions ψ^μ are anticommuting and antiperiodic,

$$\psi(\tau_1)\psi(\tau_2) = -\psi(\tau_2)\psi(\tau_1), \qquad \psi(T) = -\psi(0). \tag{9}$$

See, e.g., chapter 3 of [18] for a derivation of the path integral representations (6), (8) from quantum field theory. During the last fifteen years various efficient methods have been developed for the evaluation of this type of path integral. We are concerned here with the so-called 'string-inspired' approach [19–21] (see [18] for a review) which aims at an analytic calculation of the worldline path integral (see the contributions by GV Dunne and K Klingmüller to these proceedings for alternative approaches). This is achieved by manipulating the path integral into the Gaussian form, usually by a perturbative or higher derivative expansion, and then performing the Gaussian integration formally using worldline correlators. Those worldline Green's functions are, for the coordinate path integral,

$$\langle x^\mu(\tau_1)x^\nu(\tau_2)\rangle = -G_B(\tau_1,\tau_2)\eta^{\mu\nu}, \qquad G_B(\tau_1,\tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}, \tag{10}$$

and for the spin path integral

$$\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle = G_F(\tau_1,\tau_2)\eta^{\mu\nu}, \qquad G_F(\tau_1,\tau_2) = \text{sign}(\tau_1 - \tau_2). \tag{11}$$

We will often abbreviate $G_B(\tau_1,\tau_2) =: G_{B12}$ etc. The coordinate Green's function is not unique, since it depends on the zero mode fixing of the path integral [18]. The choice of (10) corresponds to a definition of the zero mode as the loop center of mass,

$$x_0^\mu := \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau). \tag{12}$$

To obtain, say, the one-loop N photon amplitude, one expands the Maxwell field in N plane waves with given polarization vectors ε_i^μ,

$$A^\mu(x(\tau)) = \sum_{i=1}^N \varepsilon_i^\mu e^{ik_i \cdot x(\tau)}. \tag{13}$$

This leads to each photon being represented by a photon vertex operator,

$$V_{\text{scal}}^A[k,\varepsilon] = \int_0^T d\tau\, \varepsilon \cdot \dot{x}(\tau)\, e^{ik_i \cdot x(\tau)} \qquad \text{(Scalar QED)},$$
$$V_{\text{spin}}^A[k,\varepsilon] = \int_0^T d\tau\, [\varepsilon \cdot \dot{x}(\tau) + 2i\varepsilon \cdot \psi\, k \cdot \psi]\, e^{ik_i \cdot x(\tau)} \qquad \text{(Spinor QED)}. \tag{14}$$

After expanding the interaction exponential to the *N*th order, the path integrals are Gaussian and can be evaluated by the correlators (10) and (11). This leaves one with the global proper time integral, and one parameter integral for each photon leg.

It has emerged that this formalism is particularly well suited to the calculation of QED amplitudes in a constant background field [22, 23]. The reason is that, once one has obtained the parameter integral representation for a given amplitude for the vacuum case, one can construct the corresponding integrals in the presence of a background field with constant field strength tensor $F_{\mu\nu}$ by the following simple substitutions [23, 18].

- Change the worldline Green's functions:

$$G_B(\tau_1, \tau_2) \to \mathcal{G}_B(\tau_1, \tau_2) = \frac{1}{2(eF)^2}\left(\frac{eF}{\sin(eFT)}\, e^{-ieFT\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{T}\right),$$

$$G_F(\tau_1, \tau_2) \to \mathcal{G}_F(\tau_1, \tau_2) = G_{F12}\frac{e^{-ieFT\dot{G}_{B12}}}{\cos(eFT)}. \tag{15}$$

(A 'dot' on a Green's function denotes a derivative with respect to the first variable.)

- Change the free path integral determinants

$$(4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}}\left[\frac{\sin(eFT)}{eFT}\right] \qquad (\text{Scalar } QED),$$

$$(4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}}\left[\frac{\tan(eFT)}{eFT}\right] \qquad (\text{Spinor } QED). \tag{16}$$

It should be remarked that the worldline formalism is closely related to the standard Fock–Schwinger proper-time representation of propagators in external fields [24, 25]. Therefore the resulting integral representations have generally the same structure as those obtained by that method (see, e.g., [26] and V Skalozub's contribution to these proceedings). However, the worldline approach is more global in the sense that it applies directly to a whole loop, rather than to the individual propagators making up the loop. This also implies that the worldline integral representations can be written down without fixing the ordering of the external legs along the loop. Another advantage is that the use of the so-called 'Bern–Kosower substitution rule' [20] provides a simple way of inferring the spinor loop integrands from the scalar loop ones. This effectively circumvents the usual Dirac algebra manipulations.

In flat space, the Gaussian integration of the worldline path integral can be done naively, and no ill-defined expressions are produced. For applications to gravity, we need to generalize the path integrals (6) and (8) to gravitational backgrounds. Here we enter the realm of path integrals in curved spaces, a subject notorious for its mathematical subtleties. Naively, one would introduce background gravity by a simple replacement of the kinetic term,

$$S_0 = \frac{1}{4}\int_0^T d\tau\, \dot{x}^2 \to \frac{1}{4}\int_0^T d\tau\, \dot{x}^\mu g_{\mu\nu}(x(\tau))\dot{x}^\nu. \tag{17}$$

After the usual linearization $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ this would lead to a graviton vertex operator of the form

$$\varepsilon_{\mu\nu}\int_0^T d\tau\, \dot{x}^\mu \dot{x}^\nu e^{ik\cdot x}. \tag{18}$$

However, using this vertex operator in a naive Gaussian path integration immediately leads to ill-defined expressions involving, e.g., $\delta(0)$, $\delta^2(\tau_i - \tau_j)$. A complete understanding of these difficulties, and of the steps which have to be taken to solve them in the 'string-inspired' framework, has been reached only recently [27–31]. Here we can only briefly sketch the correct procedure for the spinless case; all the necessary details and the generalization to spin half can be found in [32].

 (i) In the curved space, the path integral measure is nontrivial. Exponentiate it as follows,

$$\mathcal{D}x = Dx \prod_{0 \leqslant \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{PBC} Da\,Db\,Dc\, e^{-S_{gh}[x,a,b,c]}, \tag{19}$$

with a ghost action

$$S_{gh}[x, a, b, c] = \int_0^T \mathrm{d}\tau \, \frac{1}{4} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu). \tag{20}$$

This modifies the naive graviton vertex operator (18) to

$$V_{\mathrm{scal}}^h[k, \varepsilon] = \varepsilon_{\mu\nu} \int_0^T \mathrm{d}\tau \, [\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu] \, \mathrm{e}^{\mathrm{i}k \cdot x}. \tag{21}$$

(ii) The correlators of these ghost fields just involve δ functions,

$$\langle a^\mu(\tau_1) a^\nu(\tau_2) \rangle = 2\delta(\tau_1 - \tau_2)\eta^{\mu\nu}, \qquad \langle b^\mu(\tau_1) c^\nu(\tau_2) \rangle = -4\delta(\tau_1 - \tau_2)\eta^{\mu\nu}. \tag{22}$$

The ghost field contributions will cancel all divergent or ill-defined terms.

(iii) This cancelation of infinities leaves finite ambiguities. From the point of view of one-dimensional quantum field theory, we are dealing here with an UV divergent but super-renormalizable theory which requires only a small number of counterterms to remove all divergences. The coefficients of the counterterms have to be fixed in a way which reproduces the known spacetime physics.

(iv) As it turns out, these counterterms are regularization dependent, and in general noncovariant. This points to a violation of covariance by the regularization method. Presently, the only known covariance-preserving regularization method is one-dimensional regularization [30, 31], which has only a single covariant counterterm proportional to the curvature scalar ($-R/4$ in the present notations).

(v) The zero mode fixing leads to further subtleties. The simplest possibility would be to fix a point x_0 on the loop, $x(\tau) = x_0 + y(\tau)$. It leads to the so-called DBC ('Dirichlet boundary conditions') propagator for the coordinate field which is known to yield the same effective Lagrangian as would be obtained also by using the standard heat kernel expansion. For flat space calculations, the 'string-inspired' choice (12) is generally more convenient since it is the only one which leads to a worldline propagator for the coordinate field depending only on $\tau_1 - \tau_2$. It can be easily shown that the effective Lagrangians obtained in both ways differ only by total derivative terms [33]. This continues to be true in curved space; however here those total derivative terms turn out to be noncovariant in general, with only the DBC choice yielding a manifestly covariant form of the Lagrangian. The noncovariance of the total derivative terms present in the 'string-inspired' approach poses no problem in principle but in practice, since it invalidates the use of the Riemann normal coordinate expansion, which is an almost indispensable tool for this type of calculations. This remaining problem was solved in [34]. There it was shown that, using Riemann normal coordinates from the beginning and performing a BRST treatment of the symmetry corresponding to a shift of x_0, the difference between the two effective Lagrangians can be reduced to manifestly covariant terms. This is achieved by the addition of the further Fadeev–Popov type terms to the worldline Lagrangian in the 'string-inspired' scheme. Those terms are infinite in number but easy to determine order by order.

All this can be generalized to the spin half case [32]. Thus a standard scheme of calculation is now available for one-loop effective actions and amplitudes involving scalar or spinor loop particles and background gravitational fields. See [35] for some applications to effective actions and anomalies, [36, 37] to graviton amplitudes. More recently also worldline path integrals representing vector and antisymmetric tensor particles coupled to background gravity have been constructed [38].

3. Photon–graviton polarization tensor in a constant field

Returning to the one-loop photon–graviton amplitude in a constant electromagnetic field $F_{\mu\nu}$, we will now sketch its calculation for the scalar loop case. According to the above, this amplitude can be represented by the following expression,

$$\varepsilon_{\mu\nu}\Pi^{\mu\nu,\alpha}_{\text{scal}}(k)\varepsilon_\alpha = \frac{\text{i}e\kappa}{4}\int_0^\infty \frac{\text{d}T}{T}\,\text{e}^{-m^2 T}(4\pi T)^{-\frac{D}{2}}\det^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\left\langle V^h_{\text{scal}}[k,\varepsilon_{\mu\nu}]V^A_{\text{scal}}[-k,\varepsilon_\alpha]\right\rangle \tag{23}$$

where $\mathcal{Z}_{\mu\nu} \equiv eT F_{\mu\nu}$ and $V^{A,h}_{\text{scal}}$ are the photon and graviton vertex operators (14) and (21). Using the Wick contraction rules (10), (22) yields ($\overline{\mathcal{G}}_{B12} := \mathcal{G}_{B12} - \mathcal{G}_{B11}$, etc)

$$\Pi^{\mu\nu,\alpha}_{\text{scal}}(k) = \frac{e\kappa}{4(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{\text{d}T}{T^{1+D/2}}\text{e}^{-m^2 T}\det^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^T \text{d}\tau_1\int_0^T \text{d}\tau_2 \text{e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k}\,I^{\mu\nu,\alpha}_{\text{scal}},$$

$$I^{\mu\nu,\alpha}_{\text{scal}} = -\big(\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2\delta_{11}\eta^{\mu\nu}\big)(k\cdot\overline{\mathcal{G}}_{B12})^\alpha - \big[\ddot{\mathcal{G}}^{\mu\alpha}_{B12}(\overline{\mathcal{G}}_{B12}\cdot k)^\nu + (\mu\leftrightarrow\nu)\big]$$
$$+ (\overline{\mathcal{G}}_{B12}\cdot k)^\mu(\overline{\mathcal{G}}_{B12}\cdot k)^\nu(k\cdot\overline{\mathcal{G}}_{B12})^\alpha. \tag{24}$$

The T integral has an UV divergence at the lower limit. Using dimensional regularization, this divergence takes the form

$$\Pi^{\mu\nu,\alpha}_{\text{scal,div}}(k) = \frac{\text{i}e^2\kappa}{3(4\pi)^2}\frac{1}{D-4}C^{\mu\nu,\alpha} \tag{25}$$

where $C^{\mu\nu,\alpha}$ is the tree level vertex (3). Adding the corresponding counterterm yields the renormalized vacuum polarization tensor $\bar{\Pi}^{\mu\nu,\alpha}_{\text{scal}}$ obeying the usual renormalization condition $\bar{\Pi}^{\mu\nu,\alpha}_{\text{scal}}(k=0) = 0$. Next, appropriate photon and graviton polarizations have to be selected, where it turns out to be convenient to use the photon polarization vectors also to construct the graviton polarization tensors:

photon: $\varepsilon_\perp$, $\varepsilon_\parallel$,

graviton : $\quad \varepsilon^{\oplus\mu\nu} = \varepsilon^{\perp\mu}\varepsilon^{\perp\nu} - \varepsilon^{\parallel\mu}\varepsilon^{\parallel\nu}, \quad \varepsilon^{\otimes\mu\nu} = \varepsilon^{\perp\mu}\varepsilon^{\parallel\nu} + \varepsilon^{\parallel\mu}\varepsilon^{\perp\nu}.$

Here we have assumed a Lorentz system such that **B** and **E** is collinear, and the subscripts on the photon polarization vectors refer to the same direction. Further, no information is lost by assuming that the photon propagation is perpendicular to the field [10]. With these conventions, the components of the tree level amplitude become

$$C^{\oplus\perp} = -2B\omega, \qquad C^{\oplus\parallel} = 2E\omega, \qquad C^{\otimes\perp} = -2E\omega, \qquad C^{\otimes\parallel} = -2B\omega. \tag{26}$$

Here $\omega = k^0 = |\vec{k}|$ denotes the photon/graviton energy. Finally, it is convenient to normalize the loop amplitude by the tree level one, making the amplitude dimensionless:

$$\hat{\Pi}^{Aa}_{\text{scal}}(\hat{\omega},\hat{B},\hat{E}) \equiv \text{Re}\left(\frac{\bar{\Pi}^{Aa}_{\text{scal}}(\hat{\omega},\hat{B},\hat{E})}{-\frac{\text{i}}{2}\kappa C^{Aa}}\right) \tag{27}$$

$(A = \oplus, \otimes, a = \perp, \parallel)$. Here we have further introduced the dimensionless variables $\hat{\omega} = \frac{\omega}{m}$, $\hat{B} = \frac{eB}{m^2}$ and $\hat{E} = \frac{eE}{m^2}$.

The spinor loop calculation proceeds completely analogously, just with some additional terms coming from the evaluation of the spin path integral (8).

At this stage, the four independent components of the scalar or spinor loop amplitude are given in terms of two-parameter integrals, with integrands involving trigonometric functions of the proper times and external parameters. Let us write down here these integrals for the case of a spinor loop and a purely magnetic field:

$$\hat{\Pi}^{Aa}_{\text{spin}}(\hat{\omega},\hat{B}) = \alpha\,\text{Re}\int_0^\infty \frac{\text{d}\hat{s}}{\hat{s}}\text{e}^{-\text{i}\hat{s}}\int_0^1 \text{d}v\,\hat{\pi}^{Aa}_{\text{spin}}(\hat{s},v,\hat{\omega},\hat{B}) \tag{28}$$

$$\hat{\pi}_{\text{spin}}^{\oplus\perp} = -\frac{1}{4\pi}\left\{\frac{z}{\tanh(z)}\exp\left[z\left(\frac{\bar{A}_{B12}}{z} + \frac{1}{2}(1-v^2)\right)\frac{\hat{\omega}^2}{2\hat{B}}\right]\right.$$

$$\times\left[(S_{B12})^2 - (S_{F12})^2 + (A_{F12})^2 - \left(\bar{A}_{B12} + A_{F11}\right)\left(\bar{A}_{B12} + \frac{1}{z} + A_{F11}\right)\right.$$

$$\left.\left. + \bar{A}_{B12}((S_{B12})^2 - (S_{F12})^2 - (\bar{A}_{B12} + A_{F11})^2 + (A_{F12})^2 - vS_{B12} + S_{F12})\frac{\hat{\omega}^2}{2\hat{B}}\right] + \frac{4}{3}\right\},$$

$$\hat{\pi}_{\text{spin}}^{\otimes\|} = -\frac{1}{4\pi}\left\{\frac{z}{\tanh(z)}\exp\left[z\left(\frac{\bar{A}_{B12}}{z} + \frac{1}{2}(1-v^2)\right)\frac{\hat{\omega}^2}{2\hat{B}}\right]\right.$$

$$\left.\times\left[vS_{B12} - S_{F12} - \frac{1}{z}(\bar{A}_{B12} + A_{F11}) + \bar{A}_{B12}(vS_{B12} - S_{F12} + 1 - v^2)\frac{\hat{\omega}^2}{2\hat{B}}\right] + \frac{4}{3}\right\},$$

$$\hat{\pi}_{\text{spin}}^{\oplus\|} = 0,$$

$$\hat{\pi}_{\text{spin}}^{\otimes\perp} = 0. \tag{29}$$

Here $\hat{s} = -\mathrm{i}m^2 T$, $z = \mathrm{i}\hat{B}\hat{s}$, and the integrand involves the standard worldline coefficient functions [23].

$$S_{B12} = \frac{\sinh(zv)}{\sinh(z)},$$

$$A_{B12} = \frac{\cosh(zv)}{\sinh(z)} - \frac{1}{z},$$

$$A_{B11} = A_{B22} = \coth(z) - \frac{1}{z},$$

$$\bar{A}_{B12} = A_{B12} - A_{B11} = \frac{\cosh(zv) - \cosh(z)}{\sinh(z)}, \tag{30}$$

$$S_{F12} = \frac{\cosh(zv)}{\cosh(z)},$$

$$A_{F12} = \frac{\sinh(zv)}{\cosh(z)},$$

$$A_{F11} = A_{F22} = \tanh(z).$$

The parameter v is related to the original proper-time variables $\tau_{1,2}$ by $v = 1 - 2\tau_1/T$ (the translation invariance of the worldline correlators has been used to set $\tau_2 = 0$).

See [10] for the scalar loop and general constant field cases.

4. Properties, special cases

Let us now discuss some properties and limiting cases of the amplitude.

Ward identities: The gauge Ward identity for this amplitude gives the familiar transversality in the photon index,

$$k_\alpha \Pi^{\mu\nu,\alpha}(k) = 0. \tag{31}$$

The gravitational Ward identity, derived from invariance under infinitesimal reparametrizations, connects $\Pi^{\mu\nu,\alpha}$ with the corresponding photon–photon polarization tensor $\Pi^{\mu,\alpha}(k)$,

$$k_\mu \Pi^{\mu\nu,\alpha}(k) = \frac{\mathrm{i}}{2}\kappa F^\nu{}_\mu \Pi^{\mu,\alpha}(k). \tag{32}$$

(Similarly, non-transversality was recently found for the gluon polarization tensor in a chromomagnetic background field [39].)

Selection rules: CP invariance implies the following selection rules for the photon–graviton conversion amplitudes [3]:

- For a purely magnetic field $\varepsilon^{\oplus}$ couples only to $\varepsilon^{\perp}$ and $\varepsilon^{\otimes}$ only to $\varepsilon^{\parallel}$.
- For a purely electric field $\varepsilon^{\oplus}$ couples only to $\varepsilon^{\parallel}$ and $\varepsilon^{\otimes}$ only to $\varepsilon^{\perp}$.

This is borne out by the explicit calculation (see (26), (29)).

Pair creation thresholds: In the purely magnetic case the amplitudes are real for small ω, since the magnetic field is not capable of pair production. The pair creation thresholds ω_{cr} turn out to be identical with the ones for the corresponding photon–photon cases:

$$\hat{\omega}_{\mathrm{cr,scal}}^{\oplus\perp} = \hat{\omega}_{\mathrm{cr,scal}}^{\otimes\parallel} = 2\sqrt{1+\hat{B}}, \qquad \hat{\omega}_{\mathrm{cr,spin}}^{\oplus\perp} = 1+\sqrt{1+2\hat{B}}, \qquad \hat{\omega}_{\mathrm{cr,spin}}^{\otimes\parallel} = 2. \tag{33}$$

Calculable cases: The magnetic case is also much more amenable to an explicit calculation of the parameter integrals. In [10] we have given a detailed analysis of the following regions in parameter space (with $\mathbf{E} = 0$).

- For photon/graviton energies below threshold the parameter integrals are suitable for a straightforward numerical evaluation.
- For arbitrary ω but small B the two-parameter integrals can be reduced to one-parameter integrals over Airy functions.
- For $\omega < \omega_{\mathrm{cr}}$ and large B one finds the asymptotic behavior

$$\hat{\Pi}_{\mathrm{scal}}^{Aa}(\hat{\omega}, \hat{B}) \overset{\hat{B}\to\infty}{\sim} -\frac{\alpha}{12\pi}\ln(\hat{B}), \qquad \hat{\Pi}_{\mathrm{spin}}^{Aa}(\hat{\omega}, \hat{B}) \overset{\hat{B}\to\infty}{\sim} -\frac{\alpha}{3\pi}\ln(\hat{B}). \tag{34}$$

These leading asymptotic terms can be directly related to the corresponding UV counterterms, which is another property known from the photon–photon case [40].
- In the zero energy limit, the amplitudes relate to the magnetic Euler–Heisenberg Lagrangians $\mathcal{L}_{\mathrm{scal,spin}}^{\mathrm{EH}}(B)$.

$$\hat{\Pi}_{\mathrm{scal,spin}}^{\oplus\perp}(\hat{\omega}=0, \hat{B}) = -\frac{2\pi\alpha}{m^4}\left(\frac{1}{\hat{B}}\frac{\partial}{\partial\hat{B}} + \frac{\partial^2}{\partial\hat{B}^2}\right)\mathcal{L}_{\mathrm{scal,spin}}^{\mathrm{EH}}(\hat{B}),$$

$$\hat{\Pi}_{\mathrm{scal,spin}}^{\otimes\parallel}(\hat{\omega}=0, \hat{B}) = -\frac{4\pi\alpha}{m^4}\frac{1}{\hat{B}}\frac{\partial}{\partial\hat{B}}\mathcal{L}_{\mathrm{scal,spin}}^{\mathrm{EH}}(\hat{B}). \tag{35}$$

The identities (35) have also been derived by Gies and Shaisultanov using a different approach [41].

5. Conclusions

The calculation presented here is the first calculation of the photon–graviton vacuum polarization in a constant electromagnetic field, and also the first state-of-the-art application of the 'string-inspired' worldline formalism to an amplitude involving gravitons. Although it was not possible here to go into detail, it should be emphasized that in this formalism this calculation is only moderately more difficult than the photon–photon polarization in the field. Moreover, we have also shown that the properties of the photon–graviton polarization tensor are very similar to those of the photon–photon one. We expect that even the graviton–graviton case will be quite feasible in this formalism. In a future sequel, we intend to analyze this case at the same level of the photon–graviton one, and to study the complete one-loop photon–graviton dispersion relations (4).

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